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Degrees of Freedom of a Time Series

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We give a formal proof that if f is a smooth dynamics on a d-dimensional smooth manifold and μ is an ergodic and exact dimensional measure with Hausdorff dimension $\dim \mu > d-1$, then the number d of degrees of freedom of the dynamics can be recovered from the observation of an orbit. We implement, with this purpose, an algorithm based on the analysis of the microstructure of μ. We show how a correct estimation of d permits the computation of the Liapunov spectrum with a high accuracy avoiding the issue of the spurious exponents.

KEY WORDS: Degrees of freedom; dimension; embedology; singular value decomposition; Liapunov exponents.

1. INTRODUCTION

The phenomenon of chaos renders hopeless the exact prediction of the behaviour of some dynamical systems. It is natural in these cases to take advantage of the statistic regularity inherent in chaotic dynamics. A non-trivial problem is to determine under which conditions the main properties of an observed dynamics can be recovered from a single orbit. A limit cycle attractor in the plane provides an elementary example of a situation where the orbits of the system collapse in very few steps into a one-dimensional manifold M, making impossible the computation, for instance, of the Liapunov spectrum. One only can estimate, from the data points, the action of the tangent maps on the bundle of one-dimensional tangent spaces associated to M, from which only one Liapunov exponent can be computed. Notice that in the above example, we cannot recover either the number of state variables of the dynamics from the observation of a single orbit of the system.

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As we see below, the numerical estimation of the number of degrees of freedom of an observed dynamics in presence of strongly negative Liapunov exponents also presents a special difficulty.

The above considerations raise the problem of stating under which conditions the main properties of a dynamics (M, f, μ) where

$$\begin{cases} M & \text{is a } d\text{-dimensional smooth submanifold} \\ f\colon M \to M & \text{is a measurable dynamics} \\ \mu & \text{is an ergodic measure for } f \end{cases} \tag{1.1}$$

can be computed from a single orbit of the system.

In this paper we show that if μ is α-exact dimensional with $\alpha > d-1$, and f and M are sufficiently smooth, then we can recover the number d of degrees of freedom of the dynamics from the observation of an orbit of the system.We implement an algorithm with this purpose, and show how the estimate of d that it gives may be used in the computation of the Liapunov spectrum of the dynamics.

A measure μ is said to be α-*exact dimensional* if

$$\lim_{r \downarrow 0} \frac{\log \mu(B(\mathbf{x}, r))}{\log r} = \alpha \qquad \mu\text{-a.e.} \quad \mathbf{x} \in M.$$

This notion was introduced by Young in ref. 1 where it is proved that, if μ is α-exact dimensional, then many notions of dimension of μ, and in particular the Hausdorff dimension $\dim \mu$, coincide with α. Exact dimensional measures play nowadays a central role in Dynamical Systems. Barreira, Pesin, and Schmeling[2] have recently proved the conjecture of Eckmann and Ruelle[3] that hyperbolic measures invariant under a $C^{1+\varepsilon}$ diffeomorphism are exact dimensional.

In regard to the second condition $\alpha > d-1$, it seems to be a natural condition which ensures that the dynamics does not take place in a submanifold of dimension smaller than d, and it is satisfied by many standard dynamical systems.

Mera and Morán[4] showed that the same above conditions that permit the computation of d permit also the computation of the whole Liapunov spectrum of μ.

In Section 2 we describe the algorithm for the estimation of the number of degrees of freedom of a dynamics from a scalar time series $\{u_0, u_1, \ldots, u_{N-1}\}$ obtained from the observation of an orbit of f, i.e., $u_i = h(f^i(\mathbf{z}))$, where $\mathbf{z} \in M$ and h is a smooth unknown observable. The algorithm is based on the well known principal components analysis of the distribution of the data points in the space of m-histories. If, for increasing values of m, the estimate of the dimension of the embedded submanifold becomes stabilized to a value D, this will be our estimate of the dimension d of the submanifold where the original dynamics is defined. If the algorithm always gives m as output then two alternative hypotheses are possible: either the series is generated by a stochastic law, or it is a projection of a higher dimensional dynamics in a lower dimensional space. In both cases the hypothesis of low dimensional chaos can be rejected and it does not make sense to compute the Liapunov spectrum. We give empirical evidence on the efficiency of this algorithm for the detection of stochastic noise. This is the most likely alternative when analysing, for instance, financial markets data and, in general, social science data where, if there exists a deterministic component, it is expected to be hidden by a strong stochastic component. Techniques of noise reduction could be useful in these cases. We give also empirical evidence on the efficiency of the algorithm for the estimation of the number d of degrees of freedom of a low dimensional dynamics. The presence of low dimensional determinism has been extensively documented in the literature for the case of controlled experimental data and, more exceptionally, in some uncontrolled experimental data. In these cases a right estimation of d is a most important step for further analysis of the data.

The estimation of the number d of degrees of freedom of a smooth dynamics on a d-dimensional submanifold is specially reliable for dynamics whose Liapunov dimension[(3)] Λ is larger than $d-1$ with $\Lambda-d+1$ large enough. We also show that, for large time series, it is possible, in principle, to detect hidden dimensions linked to strongly negative Liapunov exponents, in which case $\Lambda-d+1$ is very small.

Section 3 is devoted to the proof of the theorem which gives a theoretical support to the algorithm proposed in Section 2. Finally, we show in Section 4 how to take advantage of the principal component analysis for the computation of the Liapunov exponents of a dynamics. We show how a correct estimation of the dimension d of the submanifold where the dynamics is defined may be crucial for a correct computation of the Liapunov exponents.

2. ALGORITHM FOR THE ESTIMATION OF THE NUMBER OF DEGREES OF FREEDOM OF AN OBSERVED DYNAMICS

We start assuming that (M, f, μ) satisfy the regularity conditions in (1.1), in particular we assume that f is a $C^{1+\varepsilon}$ mapping and M is a $C^{1+\varepsilon}$ submanifold. Let $\{u_0, u_1,..., u_{N-1}\}$ be a scalar time series obtained from a observation of the dynamics, i.e., $u_i = h(f^i(\mathbf{z}))$ where $\mathbf{z} \in M$ and h is an unknown smooth observable. We first use the method of local dimension analysis to determine the most likely value of d. Let $O_N^{(m)} := \{\mathbf{x}_i,\ i = 0,..., N-m\}$ be a m-dimensional embedding[5] of the time series, i.e.,

$$\mathbf{x}_i := (u_i, u_{i+1},..., u_{i+m-1}).$$

Takens[6] proved that if M is compact, and $m > 2d$, then generically $O_N^{(m)}$ is contained in a d-dimensional submanifold $J_m(M)$ of $\mathbb{R}^m$, where J_m is an embedding diffeomorphism. The same conclusion holds in the sense of prevalence under additional conditions on the dynamics.[7]

The method of local dimension explores the local structure of the empirical measure of the orbit in small balls centered at points of $O_N^{(m)}$. If $J_m(M)$ is a d-dimensional submanifold then, in small balls, the points of $O_N^{(m)}$ admit a good approximation by a d-dimensional linear subspace. The search of the k-dimensional linear subspace which best fits the data points is made through either principal component analysis or singular value analysis. For $\mathbf{x}_i \in O_N^{(m)}$ we denote by V_r the matrix which has as rows the vectors $\mathbf{x}_j - \mathbf{x}_i$, for the points $\mathbf{x}_j$ of $O_N^{(m)}$ in the closed ball $B(\mathbf{x}_i, r)$ centered at $\mathbf{x}_i$and with radius r. It is known[8] that the k-dimensional linear subspace $\mathrm{T}_{k,r}$, $k \leqslant m$, which best fits these data, in the sense that it minimizes the sum of Euclidean distances between the vectors $\mathbf{x}_j - \mathbf{x}_i$, $\mathbf{x}_j \in O_N^{(m)} \cap B(\mathbf{x}_i, r)$, and the subspace $\mathrm{T}_{k,r}$, is the linear subspace spanned by the k eigenvectors corresponding to the k largest eigenvalues of the matrix $X_r := \frac{1}{N-m+1}(V_r)^t V_r$. If we denote these eigenvalues arranged in a decreasing ordering by $\sigma_{r,j}$, $j = 1,..., m$, then the mean square error made by $\mathrm{T}_{k,r}$ is $\mathrm{E}_{r,k} := \sum_{j=k+1}^{m} \sigma_{r,j}$.

Our method determines the dimension d by studying the behaviour of the normalized error $\hat{\mathrm{E}}_{r,k} := \frac{\mathrm{E}_{r,k}}{\sum_{j=1}^{m} \sigma_{r,j}}$ as a joint function of r and k. We prove in Section 3 that if μ is an exact dimensional measure with $\dim \mu > d-1$ and f and M are $C^{1+\varepsilon}$, then for $k \geqslant d$ the normalized error $\hat{\mathrm{E}}_{r,k}$ scales as $r^{2\varepsilon}$, and for $k < d$ it goes to zero more slowly than r^a for any $a > 0$. This result gives a necessary and sufficient condition for the dimension of M to be equal to d. To get a statistically robust estimation of the value of d we average the values of $\hat{\mathrm{E}}_{r,k}$ over the points of the orbit, and we show that these averages behave as $\hat{\mathrm{E}}_{r,k}$.

The idea of studying the errors $\mathrm{E}_{r,k}$ for r fixed and different values of k was first proposed by Froehling *et al.*[9] (see ref. 9). Broomhead *et al.*[10] and Pike[11] study the scaling law of the singular values $s_{r,j} := \sqrt{\sigma_{r,j}}$, $j = 1,..., m$ of the matrix V_r as functions of r, instead of studying the errors $\mathrm{E}_{r,k}$. They gave an heuristic argument to show that if $O_N^{(m)}$ is contained in a smooth d-dimensional submanifold of $\mathbb{R}^m$ and μ is absolutely continuous w.r.t. the Lebesgue measure, then the first d singular values, after a normalization dividing by the square root of the number of points of the orbit in $B(\mathbf{x}_i, r)$, should take approximately equal values and they scale as r, whilst the last $m-d$ normalized singular values scale as r^2.

Invariant measures for a chaotic dynamics are known to display frequently geometric complexity. They often are highly anisotropic, typically supported on a fractal set. We will show in Section 3 experiments for which, in agreement with the anisotropy typical of the invariant measures, the power of the signal is not equidistributed among the first d singular values.

Moreover, in the case of a chaotic dynamics, the analysis of the individual behaviour of each singular value is not sufficient to determine the number d of degrees of freedom of the dynamics. It only can give a lower bound for d. We show below that, in order to obtain the upper bound for d, the behaviour of the normalized errors $\hat{\mathrm{E}}_{r,k}$, which aggregate the values of the $m-k$ last principal components, must be considered.

The method of principal components has been used in the literature for other related purposes as (1) to obtain optimum global coordinates, i.e., the dimension of the subspace containing the embedded manifold and not the dimension of the manifold itself,[12] (2) the estimation of a working dimension, i.e., a value of m which ensures that the observed dynamics is correctly reconstructed in the space of m-histories,[13] (3) the measurement of the noise level[14] or (4) the estimation of a local intrinsic (fractal) dimension of the attractor (refs. 15–17).

2.1. Sketch of the Theoretical Foundation of the Algorithm

We give a sketch of the proof of the theoretical basis of our algorithm. Interested readers can follow technical details in the next section. We assume that the hypotheses[6] guaranteeing that $O_N^{(m)}$ is contained in a d-dimensional $C^{1+\varepsilon}$ submanifold $J_m(M)$ hold, where J_m is an embedding $C^{1+\varepsilon}$ diffeomorphism. The normalized errors $\hat{\mathrm{E}}_{r,k}$ are, for sufficient large N, natural estimates of

$$\hat{\mathscr{E}}_r(\mathscr{T}_{k,r}) := \frac{\int_{B(\mathbf{x}_i,r)} (|\mathbf{y}-\mathbf{x}_i-\mathbf{P}_{\mathscr{T}_{k,r}}(\mathbf{y}-\mathbf{x}_i)|_2)^2 \, d\nu(\mathbf{y})}{\int_{B(\mathbf{x}_i,r)} (|\mathbf{y}-\mathbf{x}_i|_2)^2 \, d\nu(\mathbf{y})}$$

where $\mathscr{T}_{k,r}$ is the k-dimensional linear subspace which minimizes, over the set $G(n,k)$ of k-dimensional linear subspaces, the expression

$$\mathscr{E}_r(\mathrm{T}) := \int_{B(\mathbf{x}_i,r)} (|\mathbf{y}-\mathbf{x}_i-\mathbf{P}_{\mathrm{T}}(\mathbf{y}-\mathbf{x}_i)|_2)^2 \, d\nu(\mathbf{y}),$$

P_T is the orthogonal projection on $T \in G(n,k)$, and ν is the measure induced by μ under the diffeomorphism J_m. Observe that $\mathscr{E}_r(T)$ measures the L^2-distance between the orthogonal projection P_T and the identity. The linear subspace $T_{k,r}$ spanned by the k eigenvectors corresponding to the k largest eigenvalues of the matrix X_r is a natural estimate of $\mathscr{T}_{k,r}$.

We must show that $\hat{\mathscr{E}}_r(\mathscr{T}_{k,r})$ scale as $r^{2\varepsilon}$ for any $k \geqslant d$ and as $O(1)$ for $k < d$. If $k \geqslant d$ then $\mathscr{T}_{d,r} \subset \mathscr{T}_{k,r}$ so that $\mathscr{E}_r(\mathscr{T}_{k,r}) \leqslant \mathscr{E}_r(\mathscr{T}_{d,r}) \leqslant \mathscr{E}_r(\mathscr{T}_d)$ where $\mathscr{T}_d$ is the tangent space to the manifold $J_m(M)$ at $\mathbf{x}_i$. Since $J_m(M)$ is a d-dimensional $C^{1+\varepsilon}$ submanifold there is a constant K such that

$$(|\mathbf{y}-\mathbf{x}_i-\mathbf{P}_{\mathscr{T}_d}(\mathbf{y}-\mathbf{x}_i)|_2)^2 \leqslant K(|\mathbf{y}-\mathbf{x}_i|_2)^{2(1+\varepsilon)} \leqslant Kr^{2\varepsilon}(|\mathbf{y}-\mathbf{x}_i|_2)^2$$

for any $y \in B(\mathbf{x}_i, r)$ and r sufficiently small. Hence $\hat{\mathscr{E}}_r(\mathscr{T}_{k,r}) \leqslant \hat{\mathscr{E}}_r(\mathscr{T}_d) \leqslant Kr^{2\varepsilon}$ for sufficiently small r.

The most delicate part of the argument is to show that $\hat{\mathscr{E}}_r(T)$ cannot be $O(r^a)$, $a>0$, if T is a k-dimensional linear subspace with $k<d$. In fact to show this requires the stronger assumption that μ is an exact dimensional measure with $\dim \mu > d-1$. Assume on the contrary that there is a k-dimensional linear subspace T, with $k<d$, such that $\hat{\mathscr{E}}_r(T) \leqslant Kr^a$, $a>0$ for small r. Then,

$$\hat{\mathscr{E}}_r(\mathscr{T}_{d,r}) \leqslant \hat{\mathscr{E}}_r(\mathscr{T}_{k,r}) \leqslant \hat{\mathscr{E}}_r(T) \leqslant Kr^a.$$

Hence

$$\begin{aligned} &\frac{\int_{B(\mathbf{x}_i,r)} (|(\mathbf{P}_{\mathscr{T}_{k,r}}-\mathbf{P}_{\mathscr{T}_{d,r}})(\mathbf{y}-\mathbf{x}_i)|_2)^2\, d\nu(y)}{\int_{B(\mathbf{x}_i,r)} (|\mathbf{y}-\mathbf{x}_i|_2)^2\, d\nu(\mathbf{y})} \\ &= \frac{\int_{B(\mathbf{x}_i,r)} (|\mathbf{y}-\mathbf{x}_i-\mathbf{P}_{\mathscr{T}_{d,r}}(\mathbf{y}-\mathbf{x}_i)-(\mathbf{y}-\mathbf{x}_i-\mathbf{P}_{\mathscr{T}_{k,r}}(\mathbf{y}-\mathbf{x}_i))|_2)^2\, d\nu(\mathbf{y})}{\int_{B(\mathbf{x}_i,r)} (|\mathbf{y}-\mathbf{x}_i|_2)^2\, d\nu(\mathbf{y})} \\ &\leqslant \hat{\mathscr{E}}_r(\mathscr{T}_{d,r})+\hat{\mathscr{E}}_r(\mathscr{T}_{k,r}) \leqslant 2Kr^a, \end{aligned} \tag{2.1}$$

for small r. Thus, the normalized L^2-distance between the orthogonal projections $\mathbf{P}_{\mathscr{T}_{k,r}}$ and $\mathbf{P}_{\mathscr{T}_{r,d}}$ becomes very small for sufficient small r. The key point of the argument is that if the first expression in formula (2.1) goes to zero as $O(r^a)$ then the measure ν must be concentrated near $\mathscr{T}_{r,k}$, which is the kernel of $\mathbf{P}_{\mathscr{T}_{r,k}}-\mathbf{P}_{\mathscr{T}_{r,d}}$, at a speed that implies $\dim \nu \leqslant k$. Since $\dim \nu = \dim \mu > d-1$, this gives the desired contradiction.

The following example illustrates the difficulties that can arise when analyzing the local geometry of a measure through the method of principal components. Let μ be the Lebesgue measure on a planar curve in $\mathbb{R}^3$. If we apply the method of principal components to compute the bidimensional subspace which best fits the measure on a ball centered at a point of the curve, this will be, of course, the plane Π containing the curve. Perturb now slightly this measure, in such way that it spreads out on a narrow bidimensional ribbon, orthogonal to Π and containing the original curve. If this is suitably done it might occur that the plane which best fits the measure in small balls is still the plane Π, instead of the tangent plane to the bidimensional ribbon. If the dimension of the perturbed measure is larger than 1 this irregularity can only occur at exceptional points. This illustrates the role of the hypothesis $\dim \mu > d-1$ when using the method of principal components.

2.2. The Algorithm

We now see how a test, based in the above ideas, can be numerically implemented. First of all, we consider the following quantities, which measure the rate of convergence to zero of the average normalized errors $\langle \hat{\mathrm{E}}_{r,k} \rangle$ over the points $\mathbf{x}_i$ of the orbit

$$\gamma_k(r) := \frac{\ln(\langle \hat{\mathrm{E}}_{r,k} \rangle)}{\ln r} = \frac{\ln(\sum_{j=k+1}^{m} \langle \frac{\sigma_{r,j}}{\sum_{j=1}^{m} \sigma_{r,j}} \rangle)}{\ln r}, \qquad k = 1,..., m-1$$

Notice that $\gamma_k(r)$ has a physical meaning: by Birkhoff ergodic theorem[18] it is the natural estimate, for large N, of

$$\frac{\ln \int \hat{\mathscr{E}}_r(\mathscr{T}_{k,r})\, d\nu}{\ln r}.$$

We plot the points $(\ln r, \ln(\langle \hat{\mathrm{E}}_{r,k} \rangle))$ for a wide range of values of r and we estimate the rate of convergence of the normalized errors to zero as the slope γ_k of such curve. We have seen above that

$$\gamma_k = \begin{cases} 0 & \text{if} \quad k < d \\ a > 0 & \text{if} \quad k \geqslant d. \end{cases} \tag{2.2}$$

The routines for the computation of singular values display a more robust behaviour. For this reason, the eigenvalues $\sigma_{r,j}$, $j = 1,..., m$ are obtained as the squares of the singular values of the matrices $\frac{1}{\sqrt{N-m+1}} V_r$. Let $\hat{s}_{r,j} := \frac{s_{r,,j}}{\sum_{k=1}^{m} s_{r,k}}$, $j = 1,...., m$ the normalized singular values. It is easy to check that if the jth average normalized singular value $\langle \hat{s}_{r,j} \rangle$ is constant and positive for a sufficient large range of small values of r, then γ_{j-1} is null and we get directly $d \geqslant j$ from (2.2). See examples below for the practical implementation of this test.

We have implemented a FORTRAN code to obtain the dimension estimates. In order to get independence of the results from the scale of measurement, the original time series is normalized to the interval [0, 1]. The entry parameters of the code are the embedding dimension m and an initial radius $r_{\max}$. The output of the algorithm are the average normalized singular values $\langle \hat{s}_{r_i,j} \rangle$, $j = 1,..., m$, and the pairs of points $(\ln r_i, \ln(\langle \hat{E}_{r_i,k} \rangle))$, $k = 1,..., m-1$, for $r_i = \frac{r_{\max}(50-i)}{50}$, $0 \leqslant i \leqslant 50$, and for the values of i such that there are enough many neighbouring points of the data points.

Figure 1 corresponds to a scalar time series from a sample of a Uniform distribution in [0, 1]. Observe that for all the embedding dimensions m, the average normalized singular values $\langle \hat{s}_{r,j} \rangle$ are positive for any $j = 1,..., m$. Then $\gamma_k = 0$ for any $k = 1,..., m-1$, and (2.2) gives $d \geqslant m$. This gives an indication of the stochasticity of the process which generates the time series. Notice that a fast scarcity of neighbouring points for increasing values of m serves as indication that the data are not in a d-dimensional submanifold of $\mathbb{R}^m$ with $d \ll m$.

The dimensional analysis for a time series from Henon system can be seen in Fig. 2. Observe that the first two average normalized singular

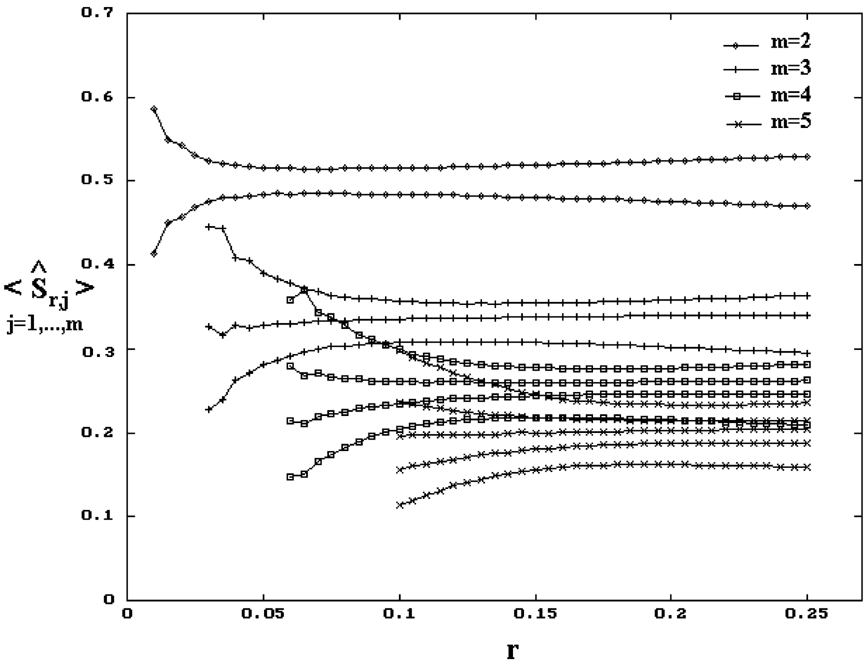

Fig. 1. Average normalized singular values as a function of the radius for a sample of a Uniform distribution. The entry parameters are $N = 50000$, $m \in \{2,..., 5\}$ and $r_{\max} = 0.25$.

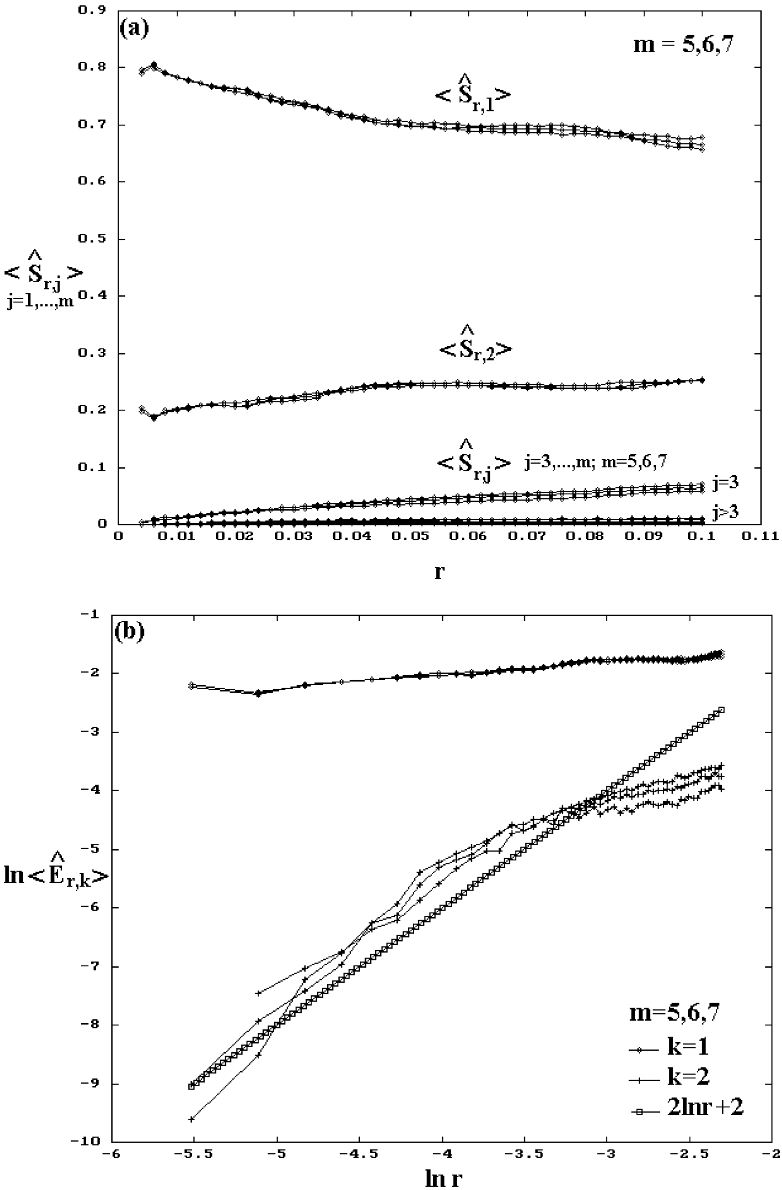

Fig. 2. Dimensional analysis for a time series from the observation of a Henon system. The equations of the dynamics are $x_{k+1} = 1 - 1.4x_k^2 + y_k$, $y_{k+1} = 0.3x_k$ and the observable is $h(x, y) = x^2 + y^2$. The entry parameters are $N = 50000$, $m \in \{5, 6, 7\}$ and $r_{\max} = 0.1$. (a) Average normalized singular values as a function of the radius. (b) Scaling law for the average normalized errors for $m \in \{5, 6, 7\}$ and $k = 1, 2$

values $\langle \hat{s}_{r,j} \rangle$, $j = 1, 2$ are positive for any $m \in \{5, 6, 7\}$, and they take remarkably stable values at any scale of observation. From this $d \geqslant 2$ follows. Notice that in agreement with the anisotropy, typical of the invariant measures, the power of the signal is not equidistributed among the first two singular values as asserted in refs. 10 and 11. The third average normalized singular value appears small for any $m \in \{5, 6, 7\}$ but we do not know if its rate of convergence to zero is sufficiently small as to guarantee that $d = 2$. For this reason we take $m > 4 = 2d^*$ where d^* is the lower bound for d obtained from the above analysis, and we plot the curves $(\ln r, \ln(\langle \hat{\mathrm{E}}_{r,k} \rangle))$ for $m \geqslant 2d^*$ and $k \in \{1, 2\}$. We can see in Fig. 2b that the slopes of the curves for $k = 1$ are null whilst they are positive for $k = 2$. Thus, we obtain the estimate $d = 2$ for the dimension of the submanifold where the observed dynamics is defined.

The case of Henon system above, perturbed with a gaussian noise of small power, can be seen in Fig. 3. It is possible to observe four strips of singular values significantly different from zero. The first two of them are

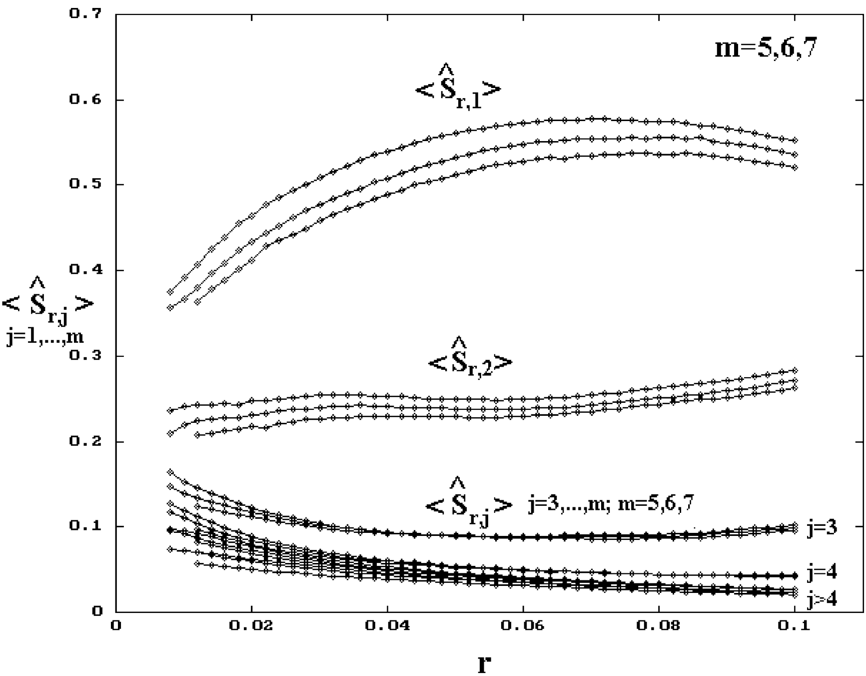

Fig. 3. Dimensional analysis for the time series in Fig. 2 perturbed with a Gaussian noise with a 1% standard deviation of the standard deviation of the unperturbed signal. The entry parameters are $N = 50000$, $m \in \{5, 6, 7\}$ and $r_{\max} = 0.1$.

bigger, and they correspond to the bidimensional local structure of the unperturbed system. The third and fourth ones, corresponding to the remaining singular values, can be identified as due to the noise, since they increase for small radii. This indicates that the noise eventually dominates the signal at small scales of observation. It is not possible to know from this analysis whether there exist more small singular values due to a higher dimensional local structure of the unperturbed system, since they would be hidden by the noise. Thus we only can assert $d \geqslant 2$. Further research, based perhaps on noise reduction techniques,[8] is needed to complete the dimensional analysis of noisy attractors.

2.3. Hidden Dimensions

Next we present the case when one or more degrees of freedom of a smooth dynamics are hidden due to the existence of strongly negative exponents which cause that the data points at small scales of observation appear as stretched along the linear span of the spatial directions corresponding to the unstable local manifold. This happen to occur for instance in Lorenz and Rössler dynamics which are three dimensional dynamics with dimension of μ close to two. From a numerical point of view, this produces a very small average normalized error made by the projections on the bidimensional linear subspaces which best fit the data points in small balls. Thus, the third average normalized singular value is small, rendering it difficult to obtain a clear indication of the existence of the third dimension from the time series.

We illustrate this fact using Lorenz dynamics (see Fig. 4). There appear three strips (Fig. 4a) around the approximate values 0.55, 0.35 and 0.1, corresponding to the averages for the first three normalized singular values. Since that corresponding to the second one is around a 35%, we can state that the dimension of the submanifold must be at least two. The third one appears to be small but it remains positive and constant even for very small radii. Thus, the estimate d of the dimension of $J_m(M)$ must be at least three for any $m \geqslant 3$. We then plot the curves $(\ln r, \ln(\langle \hat{\mathrm{E}}_{r,k} \rangle))$ for $m \in \{7, 8\}$ and $k \in \{1, 2, 3\}$. The slopes (see Fig. 4b) for $k = 1$ and $k = 2$ are null whilst for $k = 3$ are positive at least for small values of r. Then, the estimate of the dimension of $J_m(M)$ must be $d = 3$.

3. FORMAL STATEMENTS AND PROOFS

We start giving some definitions and notation. Let (M, f, μ) be a dynamics satisfying conditions (1.1). We denote by $O_N(\mathbf{z})$ the first N points

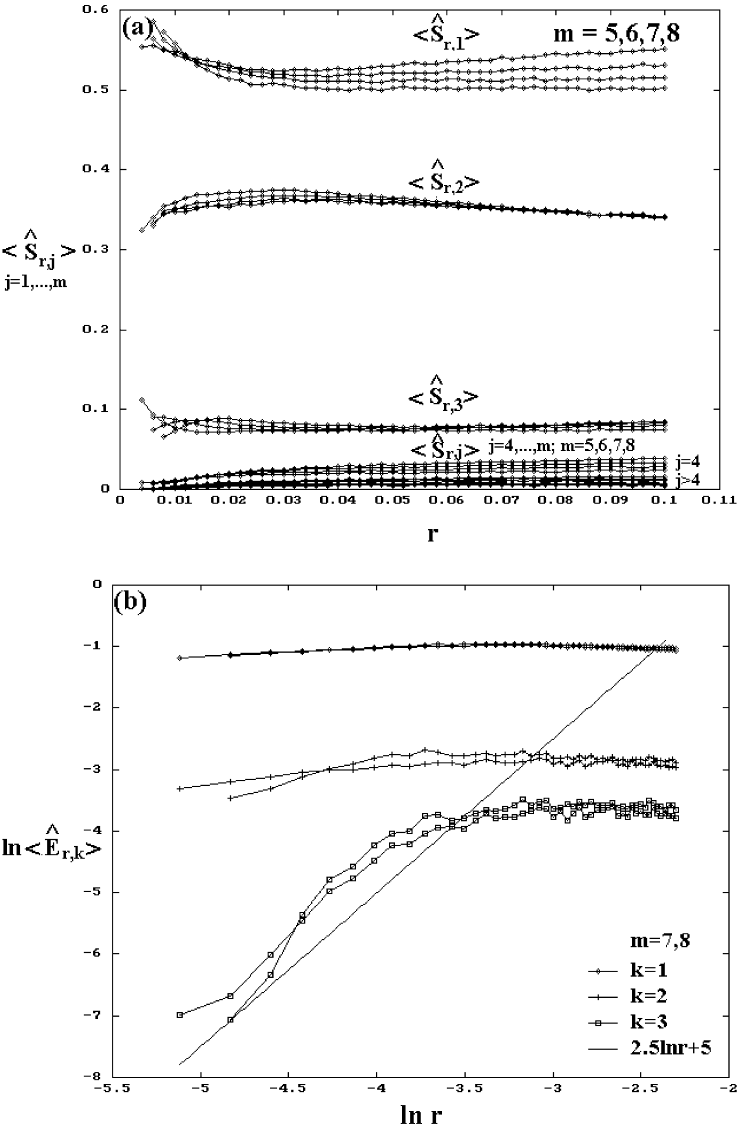

Fig. 4. Dimensional analysis for a time series from the observation of the Lorenz dynamics. The equations are $\dot{x} = -16(x - y)$, $\dot{y} = -xz + 45.92x - y$, $\dot{z} = xy - 4z$ and the observable is $h(x, y, z) = y$. A orbit of this system is obtained using a fourth-order Runge-Kutta method with a integration time step $h = 0.001$, and the sample time is $\Delta t = 0.03$. The entry parameters are $N = 50000$, $m \in \{5, 6, 7, 8\}$ and $r_{\max} = 0.1$. (a) Average normalized singular values as a function of the radius. (b) Scaling law for the average normalized errors for $m \in \{7, 8\}$, and $k = 1, 2, 3$.

of the orbit of $\mathbf{z} \in M$, and by μ_N the corresponding empirical measure of the orbit, that is $\mu_N := \frac{1}{N} \sum_{i=0}^{N-1} \delta_{f^i(\mathbf{z})}$. The weak convergence of a sequence of measures $\{\nu_N\}$ to a measure ν is denoted by $\nu_N \xrightarrow{w} \nu$, and the support of a measure ν by $spt(\nu)$. We denote by $\nu \mid A$ the restriction of the measure ν to the set A, by ∂A the boundary of the set A, and by $g_\# \nu$ the measure induced by ν under the mapping g, that is $g_\# \nu(A) = \nu(g^{-1}(A))$ for any set $A \subset M$.

Let $\dim(A)$ denote the Hausdorff dimension[19] of the set A. The *Hausdorff dimension* of the measure ν is

$$\dim \nu := \inf\{\dim(A) \colon \mu(A) > 0\}.$$

The set of k-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $G(n, k)$, and the orthogonal projection of $\mathbb{R}^n$ onto $\mathrm{T} \in G(n, k)$ is denoted by P_{T}.

In this section we give conditions guaranteeing that the number d of degrees of freedom of the dynamics can be recovered from $O_N(\mathbf{z})$. Notice that this framework covers the case of reconstructed dynamics. The m-dimensional embedding of the time series $O_N^{(m)}$ (see Section 1 for the notation) is an orbit of the dynamics $(J_m(M), f^* := J_m \circ f \circ J_m^{-1}, \nu := f^*_\# \mu)$ where $J_m(\mathbf{z}) := (h(\mathbf{z}), h(f(\mathbf{z})), \ldots, h(f^{m-1}(\mathbf{z})))$.

Theorem 3.1. Let M be a $C^{1+\varepsilon}$ d-dimensional submanifold of $\mathbb{R}^n$, let f be a measurable dynamics on M, and let μ be an f-invariant, ergodic and α-exact dimensional Borel probability measure on M with $\alpha > d-1$. Let $\mathbf{z} \in M$, $\mathbf{x}_i \in O_N(\mathbf{z})$, $r > 0$, and $X_{N,r} := \frac{1}{N} (V_{N,r})^t \, V_{N,r}$, where $V_{N,r}$ is the matrix which has as rows the coordinates of the vectors $\mathbf{x}_j - \mathbf{x}_i$ with respect to an arbitrarily chosen orthonormal basis of $\mathbb{R}^n$, with the index j ranging in the set $\mathcal{N} := \{j \colon \mathbf{x}_j \in O_N(\mathbf{z}) \cap B(\mathbf{x}_i, r)\}$. Let $\sigma_{N,r,1} \geqslant \cdots \geqslant \sigma_{N,r,n}$ be the eigenvalues of $X_{N,r}$,

$$\hat{\sigma}_{N,r,j} := \frac{\sigma_{N,r,j}}{\sum_{l=1}^{n} \sigma_{N,r,l}}, \qquad j = 1, \ldots, n, \quad \text{and}$$

$$\delta_k := \liminf_{r \downarrow 0} \lim_{N \to \infty} \frac{\ln(\sum_{j=k+1}^{n} \hat{\sigma}_{N,r,j})}{\ln r}, \qquad k = 0, \ldots, n-1.$$

Then the following implications hold μ-a.e. $\mathbf{z}$,

$$k \geqslant d \Leftrightarrow \delta_k > 0.$$

To prove this theorem we need the following lemma.

Lemma 3.2. Let μ be a Borel probability measure on $M \subset \mathbb{R}^n$. Let $\mathbf{x} \in spt(\mu)$, $k \in \{0, 1,..., n-1\}$ and let $\mathcal{T}_{k,r} \in G(n, k)$ be the subspace for which the minimum, over the set $G(n, k)$, of

$$\mathcal{E}_r(\mathrm{T}) := \int_{B(\mathbf{x}, r)} (|\mathbf{y}-\mathbf{x}-\mathrm{P_T}(\mathbf{y}-\mathbf{x})|_2)^2 \, d\mu(\mathbf{y}),$$

is attained, where $\mathrm{P_T}$ denotes the orthogonal projection of $\mathbb{R}^n$ onto $\mathrm{T} \in G(n, k)$. Let B be an arbitrary orthonormal basis of $\mathbb{R}^n$, and let X_r be the matrix with (i, j) entry given by

$$\int_{B(\mathbf{x}, r)} (y_i - x_i)(y_j - x_j) \, d\mu(\mathbf{y}),$$

where $y_i - x_i$ is the ith coordinate of the vector $\mathbf{y}-\mathbf{x}$ with respect to B. Let $\sigma_{r,1} \geqslant \cdots \geqslant \sigma_{r,n}$ be the eigenvalues of X_r, and let $\mathbf{w}_{r,i}$, $i = 1,..., n$ be the corresponding eigenvectors. Then

(i) $\mathcal{T}_{k,r} = span\{\mathbf{w}_{r,1},..., \mathbf{w}_{r,k}\}$.

(ii) $\mathcal{E}_r(\mathcal{T}_{k,r}) = \sum_{j=k+1}^{n} \sigma_{r,j}$.

Proof of Lemma 3.2. See the proof of Theorem 2 given in ref. 8 for discrete measures.

Remark 1. Notice that, for $k = 0$, Lemma 3.2 gives

$$\sum_{j=1}^{n} \sigma_{r,j} = \int_{B(\mathbf{x}, r)} (|\mathbf{y}-\mathbf{x}|_2)^2 \, d\mu(\mathbf{y}).$$

Proof of Theorem 3.1. Let $\mathrm{T}_{N,k,r}$ be the subspace in $G(n, k)$ for which the minimum of

$$\mathrm{E}_{N,r}(\mathrm{T}) := \int_{B(\mathbf{x}_i, r)} (|\mathbf{y}-\mathbf{x}_i-\mathrm{P_T}(\mathbf{y}-\mathbf{x}_i)|_2)^2 \, d\mu_N(\mathbf{y})$$

is attained (recall that μ_N is the empirical measure of the orbit). It is known (see ref. 8 or Lemma 3.2) that

$$\mathrm{E}_{N,r}(\mathrm{T}_{N,k,r}) = \sum_{j=k+1}^{n} \sigma_{N,r,j} \qquad \text{and} \qquad \mathrm{T}_{N,k,r} = span\{\mathbf{w}_{N,r,1},..., \mathbf{w}_{N,r,k}\},$$

where $\mathbf{w}_{N,r,j}$, $j = 1,..., k$ are the eigenvectors corresponding to the first k eigenvalues of $X_{N,r}$.

Let X_r be the matrix in Lemma 3.2 for the point $\mathbf{x}_i$ and the measure μ, let $\sigma_{r,1} \geqslant \cdots \geqslant \sigma_{r,n}$ be the eigenvalues of X_r, and let $\{\mathbf{w}_{r,1},..., \mathbf{w}_{r,n}\}$ be the corresponding eigenvectors. Using that M is a d-dimensional submanifold, $\partial B(\mathbf{x}_i, r)$ is an $n-1$ dimensional submanifold, and dim $\mu > d-1$, we obtain that $\mu(\partial B(\mathbf{x}_i, r) \cap M) = 0$ for enough small r. This fact, together with $\mu_N \xrightarrow{w} \mu$ for μ-a.e. $\mathbf{z} \in M$, gives $\mu_N \mid B(\mathbf{x}_i, r) \xrightarrow{w} \mu \mid B(\mathbf{x}_i, r)$. Therefore $\lim_{N\to\infty} X_{N,r} = X_r$ for μ-a.e. $\mathbf{z} \in M$ and any $\mathbf{x}_i \in O_N(\mathbf{z})$. Then, by the continuous dependence of the spectrum of a matrix upon its entries and Lemma 3.2 we have

$$\lim_{N\to\infty} \sum_{j=k+1}^{n} \hat{\sigma}_{N,r,j} = \lim_{N\to\infty} \sum_{j=k+1}^{n} \frac{\sigma_{N,r,j}}{\sum_{l=1}^{n} \sigma_{N,r,l}} = \sum_{j=k+1}^{n} \frac{\sigma_{r,j}}{\sum_{l=1}^{n} \sigma_{r,l}}$$

$$= \frac{\int_{B(\mathbf{x}_i, r)} (|\mathbf{y}-\mathbf{x}_i - \mathbf{P}_{\mathscr{T}_{k,r}}(\mathbf{y}-\mathbf{x}_i)|_2)^2 \, d\mu(\mathbf{y})}{\int_{B(\mathbf{x}_i, r)} (|\mathbf{y}-\mathbf{x}_i|_2)^2 \, d\mu(\mathbf{y})} \quad \mu\text{-a.e.} \quad \mathbf{z} \in M,$$

where $\mathscr{T}_{k,r} = span\{\mathbf{w}_{r,1},..., \mathbf{w}_{r,k}\}$. Therefore

$$\delta_k = \liminf_{r\downarrow 0} \frac{\ln\left(\frac{\mathscr{E}_r(\mathscr{T}_{k,r})}{\int_{B(\mathbf{x}_i, r)} (|\mathbf{y}-\mathbf{x}_i|_2)^2 \, d\mu(\mathbf{y})}\right)}{\ln r} \quad \mu\text{-a.e.} \quad \mathbf{z} \in M.$$

(i) We prove that $k \geqslant d$ implies $\delta_k > 0$.

Let (U, ϕ) be a chart at $\mathbf{x}_i \in M$, where U is a neighborhood of $\mathbf{x}_i$, and ϕ is a $C^{1+\varepsilon}$ diffeomorphism on U such that $\phi(\mathbf{x}_i) = \mathbf{0}$. Since ϕ is $C^{1+\varepsilon}$ there exist constants L and r_0 such that $|\phi^{-1}(\mathbf{t}) - \phi^{-1}(\mathbf{0}) - D\phi^{-1}(\mathbf{0})\, \mathbf{t}|_2 \leqslant L(|\mathbf{t}|_2)^{1+\varepsilon}$ holds if $|\mathbf{t}|_2 < r_0$. Furthermore for any constant K, with $K > \|D\phi(\mathbf{x}_i)\|_2$ there exists an $r_1 < \frac{r_0}{K}$ such that $|\phi(\mathbf{y}) - \phi(\mathbf{x}_i)|_2 \leqslant K\, |\mathbf{y}-\mathbf{x}_i|_2$ holds if $|\mathbf{y}-\mathbf{x}_i|_2 \leqslant r_1$. Let $\mathscr{T}_{\mathbf{x}_i}(M)$ denote the tangent space of M at $\mathbf{x}_i$. Let $r < r_1$, $\mathbf{y} \in B(\mathbf{x}_i, r)$ and $\mathbf{t} = \phi(\mathbf{y})$. Then

$$\begin{aligned} |\mathbf{y}-\mathbf{x}_i - \mathbf{P}_{\mathscr{T}_{\mathbf{x}_i}(M)}(\mathbf{y}-\mathbf{x}_i)|_2 &= |\phi^{-1}(\mathbf{t}) - \phi^{-1}(\mathbf{0}) - \mathbf{P}_{\mathscr{T}_{\mathbf{x}_i}(M)}(\phi^{-1}(\mathbf{t}) - \phi^{-1}(\mathbf{0}))|_2 \\ &\leqslant |\phi^{-1}(\mathbf{t}) - \phi^{-1}(\mathbf{0}) - D\phi^{-1}(\mathbf{0})\, \mathbf{t}|_2 \leqslant L(|\mathbf{t}|_2)^{1+\varepsilon} \\ &= L(|\phi(\mathbf{y}) - \phi(\mathbf{x}_i)|_2)^{1+\varepsilon} \leqslant LK^{1+\varepsilon} r^{\varepsilon}\, |\mathbf{y}-\mathbf{x}_i|_2, \end{aligned}$$

where the first inequality holds because $D\phi^{-1}(\mathbf{0})\, \mathbf{t}$ is a vector in $\mathscr{T}_{\mathbf{x}_i}(M)$. Thus $\mathscr{E}_r(\mathscr{T}_{d,r}) \leqslant \mathscr{E}_r(\mathscr{T}_{\mathbf{x}_i}(M)) \leqslant (LK^{1+\varepsilon} r^{\varepsilon})^2 \int_{B(\mathbf{x}_i, r)} (|\mathbf{y}-\mathbf{x}_i|_2)^2 \, d\mu(\mathbf{y})$ for $r < r_1$, and therefore $\delta_d \geqslant 2\varepsilon$ holds. If $k \geqslant d$ then $\mathscr{E}_r(\mathscr{T}_{k,r}) \leqslant \mathscr{E}_r(\mathscr{T}_{d,r})$ so that $\delta_k \geqslant \delta_d \geqslant 2\varepsilon$.

(ii) We prove that $\delta_k > 0$ implies that $k \geqslant d$.

Assume that $k < d$. Since $\delta_k > 0$, for any η with $0 < \eta < \delta_k$ there is an r_0 such that

$$\mathscr{E}_r(\mathscr{T}_{d,r}) \leqslant \mathscr{E}_r(\mathscr{T}_{k,r}) \leqslant r^{\delta_k - \eta} \int_{B(\mathbf{x}_i, r)} (|\mathbf{y} - \mathbf{x}_i|_2)^2 \, d\mu(\mathbf{y}) \tag{3.1}$$

for $r < r_0$. Since μ is an α-exact dimensional measure, with $\alpha > d-1$, we can obtain an analogous result to that given in Theorem 2 of ref. 20: for any $\tau > 0$, and for μ-a.e. $\mathbf{x}_i \in M$ there are positive constants C, S and r_1, with $S < 1$ and $r_1 < r_0$, depending on $\mathbf{x}_i$ and on the chosen atlas of M, such that for any linear map β: $\mathbb{R}^n \to \mathbb{R}^n$,

$$\|\beta\|_2 \leqslant \frac{C}{r^{1+\tau}} \left[\frac{1}{\mu(B(\mathbf{x}_i, Sr))} \int_{B(\mathbf{x}_i, r)} (|\beta(\mathbf{y} - \mathbf{x}_i)|_2)^2 \, d\mu(\mathbf{y}) \right]^{1/2} \tag{3.2}$$

holds for $r < r_1$. Taking τ with $0 < \tau < \frac{\delta_k - \eta}{2}$, $\beta = \mathrm{P}_{\mathscr{T}_{d,r}} - \mathrm{P}_{\mathscr{T}_{k,r}}$ in (3.2), and using (3.1) we obtain

$$\begin{aligned} \|\mathrm{P}_{\mathscr{T}_{d,r}} - \mathrm{P}_{\mathscr{T}_{k,r}}\|_2 &\leqslant \frac{C}{r^{1+\tau}} \left[\frac{1}{\mu(B(\mathbf{x}_i, Sr))} \int_{B(\mathbf{x}_i, r)} (|(\mathrm{P}_{\mathscr{T}_{d,r}} - \mathrm{P}_{\mathscr{T}_{k,r}})(\mathbf{y} - \mathbf{x}_i)|_2)^2 \, d\mu(\mathbf{y}) \right]^{1/2} \\ &\leqslant \frac{C}{r^{1+\tau} (\mu(B(\mathbf{x}_i, Sr)))^{\frac{1}{2}}} [(\mathscr{E}_r(\mathscr{T}_{k,r}))^{\frac{1}{2}} + (\mathscr{E}_r(\mathscr{T}_{d,r}))^{\frac{1}{2}}] \\ &\leqslant 2C r^{\frac{\delta_k - \eta}{2} - \tau} \left(\frac{\mu(B(\mathbf{x}_i, r))}{\mu(B(\mathbf{x}_i, Sr))} \right)^{1/2} \end{aligned} \tag{3.3}$$

for $r < r_1$. This inequality, together with the fact that μ is an exact dimensional measure, give that $\lim_{r \to 0} \|\mathrm{P}_{\mathscr{T}_{d,r}} - \mathrm{P}_{\mathscr{T}_{k,r}}\|_2 = 0$, which contradicts that $\mathscr{T}_{k,r} \in G(n, k)$ with $k < d$. ▮

Remark 2. The hypotheses that μ is an α-exact dimensional measure with $\dim \mu > d-1$ forces the first inequality in (3.3) to be true, which gives the desired contradiction. In fact, the two last inequalities in (3.3) might hold for measures concentrated near the kernel $\mathscr{T}_{k,r}$ of the linear map $\mathrm{P}_{\mathscr{T}_{d,r}} - \mathrm{P}_{\mathscr{T}_{k,r}}$, but then the dimension of such measures has to be less than or equal to $d-1$.

Remark 3. Let $\mathbf{x}_i \in O_N(\mathbf{z})$ and let $\sigma^{(i)}_{N,r,j}$, $j = 1, ..., n$ be the eigenvalues, given in a decreasing ordering, of the matrix $X_{N,r}$ at $\mathbf{x}_i$. Let I be a subset of indices of $\{1,...., N\}$, and let $\langle\hat{\sigma}_{N,r,j}\rangle$ be an average, over the points corresponding to the indices in I, of the normalized jth eigenvalue, $j = 1,..., n$. Let

$$\gamma_{l,I} := \liminf_{r\downarrow 0} \lim_{N\to\infty} \frac{\ln(\sum_{j=l+1}^{n} \langle\hat{\sigma}_{N,r,j}\rangle)}{\ln r}, \qquad l = 0,..., n-1.$$

Then by the inequalities

$$\min\{\sigma_{N,r,j}^{(i)} : i \in I\} \leqslant \langle\hat{\sigma}_{N,r,j}\rangle \leqslant \max\{\sigma_{N,r,j}^{(i)} : i \in I\}$$

and Theorem 3.1 we have that

$$l \geqslant d \Leftrightarrow \gamma_{l,I} > 0$$

for μ-a.e. $\mathbf{z} \in M$, which also implies $\langle\hat{\sigma}_{N,r,j}\rangle \sim 0$ and $\langle\hat{s}_{N,r,j}\rangle \sim 0$ for any $j > d$, large N, and small r, where $\hat{s}_{N,r,j} := \frac{s_{N,r,j}}{\sum_{j=1}^{n} s_{N,r,j}}$ and $s_{N,r,j} = \sqrt{\sigma_{N,r,j}}$ is the jth singular value of the matrix $\frac{1}{\sqrt{N}} V_{N,r}$. This gives a theoretical support to the algorithm proposed in Section 2.

4. ADAPTATION OF THE ECKMANN AND RUELLE ALGORITHM TO THE COMPUTATION OF THE LIAPUNOV EXPONENTS IN SMOOTH SUBMANIFOLDS

Let (M, f, μ) be a dynamics satisfying the regularity conditions in (1.1). We assume that f is a $C^{1+\varepsilon}$ mapping and M is a d-dimensional $C^{1+\varepsilon}$ manifold. Let $\{u_0, u_1,..., u_{N-1}\}$ be a scalar time series obtained from a smooth observation of the dynamics, and let $O_N^{(m)} := \{\mathbf{x}_i,\ i = 0,..., N-m\}$ be a m-dimensional embedding of the time series with $m > 2d$. The Eckmann and Ruelle algorithm (E.R.A. for the sequel) for the estimation of the Liapunov spectrum (see ref. 21) is based on the estimation of the tangent maps from $O_N^{(m)}$. They take as estimate of the tangent map at $\mathbf{x}_i$ the linear map S_i which minimizes, in the set $\mathscr{L}(\mathbb{R}^m)$ of linear maps $S: \mathbb{R}^m \to \mathbb{R}^m$, the mean square error

$$\sum_{j\in\mathscr{N}} (|\mathbf{x}_{j+1} - \mathbf{x}_{i+1} - S(\mathbf{x}_j - \mathbf{x}_i)|_2)^2,$$

where $\mathscr{N}$ denotes the set of indices corresponding to a given number of closest neighbouring points to $\mathbf{x}_i$.

Since $O_N^{(m)} \subset J_m(M)$, where $J_m(M)$ is a d-dimensional submanifold of $\mathbb{R}^m$, the tangent map at $\mathbf{x}_i$ for the embedded dynamics is defined on the space $\mathrm{T}_{\mathbf{x}_i}(J_m(M))$ tangent to $J_m(M)$ at $\mathbf{x}_i$. Therefore, we can avoid the issue of the detection of the $m-d$ spurious exponents (compare the method used in refs. 22 and 23 to solve the issue of the spurious exponents), if the estimates of the tangent maps belong to $\mathscr{L}(\mathbb{R}^d)$. The procedure we describe below was first proposed by Darbyshire and Broomhead[24] and Stoop and Parisi.[25]

Let T_i^* be an estimate of $\mathrm{T}_{\mathbf{x}_i}(J_m(M))$, and, for $j \in \mathscr{N}$, let $\mathrm{P}_{\mathrm{T}_i^*}(\mathbf{x}_j - \mathbf{x}_i)$, be the orthogonal projection of the vector $\mathbf{x}_j - \mathbf{x}_i$ on T_i^* and let r be a small radius. We give as estimate of the tangent map at $\mathbf{x}_i$ the linear map $S_{N,r,i}$ which best describes how the evolution law takes the vectors $\mathrm{P}_{\mathrm{T}_i^*}(\mathbf{x}_j - \mathbf{x}_i)$ to the vectors $\mathrm{P}_{\mathrm{T}_{i+1}^*}(\mathbf{x}_{j+1} - \mathbf{x}_{i+1})$, for $j \in \mathscr{Q} := \{ j \in \mathscr{N} : \mathrm{P}_{\mathrm{T}_i^*}(\mathbf{x}_j - \mathbf{x}_i) \in B(\mathbf{0}, r)\}$. That is, $S_{N,r,i}$ is the linear map which minimizes, in $\mathscr{L}(\mathbb{R}^d)$, the mean square error

$$\mathbb{E}_{N,r,i}(S) := \sum_{j \in \mathscr{Q}} (|\mathrm{P}_{\mathrm{T}_{i+1}^*}(\mathbf{x}_{j+1} - \mathbf{x}_{i+1}) - S\mathrm{P}_{\mathrm{T}_i^*}(\mathbf{x}_j - \mathbf{x}_i)|_2)^2.$$

We take as estimate of $\mathrm{T}_{\mathbf{x}_i}(J_m(M))$ the d-dimensional linear subspace T_i^* which best fits the data, in the sense that it minimizes the sum of the Euclidean distances between the vectors $\mathbf{x}_j - \mathbf{x}_i$ and T_i^*, for $j \in \mathscr{N}$. T_i^* is the linear subspace spanned by the d eigenvectors corresponding to the d largest eigenvalues of the correlation matrix X_{r_0} (see Section 2 for the definition of this matrix). Let $\mathbf{G}_i$ be the orthonormal basis of T_i^* given by these eigenvectors, and let B_i be the $d \times m$ matrix whose rows are the coordinates of the vectors of $\mathbf{G}_i$ expressed in the canonical basis of $\mathbb{R}^m$. Then, B_i is the matrix of the orthogonal projection $\mathrm{P}_{\mathrm{T}_i^*} : \mathbb{R}^m \to \mathrm{T}_i^*$ expressed with respect to the canonical basis of the original space $\mathbb{R}^m$ and with respect to the orthonormal basis $\mathbf{G}_i$ of the image space T_i^*. Therefore, the matrix of the linear map $S_{N,r,i}$ expressed with respect to the orthonormal bases $\mathbf{G}_i$ and $\mathbf{G}_{i+1}$ is the $d \times d$ matrix which minimizes the expression

$$\mathbb{E}_{N,r,i}(S) = \sum_{j \in \mathscr{Q}} (|B_{i+1}(\mathbf{x}_{j+1} - \mathbf{x}_{i+1}) - SB_i(\mathbf{x}_j - \mathbf{x}_i)|_2)^2. \tag{4.1}$$

Let λ_j, $j = 1,..., d$ be the Liapunov exponents of the tangent map. Let $\alpha_{K,N,r,j}$, $j = 1,..., d$ be the estimates of the Liapunov exponents provided by the algorithm, obtained from an iterative QR decomposition of the $d \times d$ matrix $S_{N,r,K-1} \cdot S_{N,r,K-2} \cdots S_{N,r,1} \cdot S_{N,r,0}$. In ref. 4 it is proved that if the measure μ is exact dimensional with $\dim \mu > d-1$, then the adaptation of E.R.A. described above gives the whole Liapunov spectrum of the tangent map, i.e.,

$$\lim_{K \to \infty} \lim_{r \to 0} \lim_{N \to \infty} \alpha_{K,N,r,j} = \lambda_j, \qquad j = 1,..., d.$$

The proof relies on the convergence of $S_{N,r,i}$ to $Df^*(\mathbf{x}_i)$ where $f^* = J_m \circ f \circ J_m^{-1}$, that is

$$\lim_{r \to 0} \lim_{N \to \infty} S_{N,r,i} = Df^*(\mathbf{x}_i).$$

In ref. 4 we show that $\lim_{N \to \infty} S_{N,r,i} = S_{r,i}$ where $S_{r,i}$ is the natural estimate of the action of Df^* in the ball $B(\mathbf{x}_i, r)$, in the sense that it is the linear map which minimizes errors, with respect to the ergodic measure $v = J_m \# \mu$. The most delicate part of the proof is to see that, under the above hypotheses, $\lim_{r \to 0} S_{r,i} = Df^*(\mathbf{x}_i)$ holds. Notice that if the ergodic measure is concentrated near a submanifold with dimension less than or equal to d that convergence may fail to hold. This actually occurs, for instance, in the case of a limit cycle attractor in the plane, mentioned at the beginning of the introduction.

If the dimension d is correctly estimated and the data are consistent with a differentiable dynamics (a test of differentiability can be seen in ref. 26) then the whole Liapunov spectrum of the dynamics can be computed with an arbitrary accuracy (see Table I).

A right estimation of the true value of d is essential for a right computation of the Liapunov spectrum. If d is overestimate there appear spurious exponents. If d is underestimate then the algorithm does not correctly work (see Table II).

If the estimate of d coincides with the dimension of the unstable manifold, the algorithm computes the action of the tangent map along the unstable manifold. One might hope that in this case the algorithm should give the non negative part of the Liapunov spectrum. This seems in contradiction with the numerical results in Table II. The reason of this disagreement could be due to the fact that the unstable global manifold is

Table I. Liapunov Exponents of Lorenz Dynamics in Fig. 4 as a Function of the Embedding Dimension m[a] for $D=3$

	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
α_1	3.7285	1.5453	1.5121	1.5233	1.5117	1.5010
α_2	−0.6885	−0.0604	0.0013	−0.0192	−0.0094	0.0012
α_3	−25.3491	−25.5362	−24.6822	−23.8908	−22.6678	−22.4483

[a] The data are the observation of a 500000 points orbit of this system with observable $h(x, y, z) = y$. We have taken $D = 3$ as the estimate of the dimension d of the submanifold where the dynamics is defined. It is known that the true values of the Liapunov exponents are $\lambda_1 \sim 1.5$, $\lambda_2 = 0$ and $\lambda_3 \sim -22.5$. Notice the high accuracy of the estimates of the three exponents for $m \geqslant 7$

Table II. Liapunov Exponents of Lorenz Dynamics in Fig. 4 as a Function of the Embedding Dimension m^a for $D=2$

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
α_1	25.3697	7.0058	4.7285	2.7706	2.2330	1.8587	1.6733
α_2	−5.1752	−5.3109	−3.4121	−1.9610	−1.2353	−0.7798	−0.4810
	$m=9$	$m=10$	$m=11$	$m=12$	$m=13$	$m=14$	$m=15$
α_1	1.5757	1.5968	1.5457	1.5074	1.4688	1.5490	1.4770
α_2	−0.3888	−0.3486	−0.2178	−0.1691	−0.1099	−0.1959	−0.1041

[a] The data are the observation of a 100000 points orbit of this system with observable $h(x, y, z) = y$. We have taken $D=2$ as the estimate of the dimension d of the submanifold where the dynamics is defined. A non correct estimate of the dimension d cause a bad estimate for the first Liapunov exponent for $m \leqslant 8$ and for the second one for $m \leqslant 11$

not a differentiable manifold because of the complex folds of the global unstable manifold onto itself over the chaotic attractor. This explanation is also confirmed by Table II, which shows that for increasing embedding dimensions, for which the unstable manifold is supposed to be unfolded, the algorithm gives reasonably good estimates of the non negative part of the Liapunov spectrum. If this pattern is observed in computing the Liapunov spectrum of a dynamics, it gives an indication that the estimate D of the number of degrees of freedom d should be increased.

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REFERENCES

1. L. S. Young, *Ergod. Th. & Dynam. Sys.* **2**:109 (1982).
2. L. Barreira, Y. Pesin, and J. Schmeling, *Ann. of Math.* **149**:755 (1999).
3. J.-P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **57**:617 (1985).
4. M. E. Mera and M. Morán, *Ergodic Theory Dynam. Systems* **20**:531 (2000).
5. N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw, *Phys. Rev. Lett.* **45**:712 (1980).
6. F. Takens, in *Dynamical Systems and Turbulence*, D. A. Rang and L. S. Young, eds., Lectures Notes in Mathematics, Vol. 898, p. 396 (1981).
7. T. Sauer, J. A. York, and M. Casdagli, *J. Statist. Phys.* **65**:579 (1992).
8. R. Cawley and Guan-Hsong Hsu, *Phys. Rev. A* **46**:3057 (1992).
9. H. Froehling, J. P. Crutchfield, D. Farmer, N. H. Packard, and R. Saw, *Physica D* **3**:605 (1981).

10. D. S. Broomhead, R. Jones, and G. P. King, *J. Phys. A Math. Gen.* **20**:L563 (1987).
11. E. R. Pike, *Computational Systems. Natural and Artificial*, H. Haken, ed., Vol. 86 (Springer-Verlag, Berlín, 1987).
12. P. King, R. Jones, and D. S Broomhead, *Nucl. Phys. B* **2**:379 (1987)
13. A. I. Mees, P. E. Rapp, and L. S. Jennings, *Phys. Rev. A* **36**:340 (1987).
14. D. S. Broomhead and G. P. King, *Physica D* **20**:217 (1986).
15. A. Pasamante, T. Hediger, and M. Gollub, *Phys. Rev. A* **39**:3640 (1989).
16. T. Hediger, A. Pasamante, and M. E. Farrell, *Phys. Rev. A* **41**:5325 (1990).
17. T. M. Buzug, J. von Stamm, and G. Pfister, *Phys. Lett. A* **202**:183 (1995).
18. P. Walters, *An Introduction to Ergodic Theory*, (Springer-Verlag, 1982).
19. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability* (Cambridge University Press, 1995).
20. M. E. Mera and M. Morán, *J. Math. Anal. Applic.* **235**:454 (1999).
21. J.-P. Eckmann, S. O. Kamphorst, D. Ruelle, and S. Ciliberto, *Phys. Rev. A* **34**:4971 (1986).
22. R. Brown, P. Bryan, and H. D. I. Abarbanel, *Phys. Rev. A* **43**:2787 (1991).
23. U. Parlitz, *Int. J. Bif. Chaos* **2**:155 (1992).
24. A. G. Darbyshire and D. S. Broomhead, *Physica D* **89**:287 (1996).
25. R. Stoop and J. Parisi, *Physica D* **50**:89 (1991).
26. M. E. Mera and M. Morán, Degree of differentiability of an observed dynamics, working paper.